

SCHRÖDINGER TYPE PROPAGATORS, PSEUDODIFFERENTIAL OPERATORS AND MODULATION SPACES

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ABSTRACT. We prove continuity results for Fourier integral operators with symbols in modulation spaces, acting between modulation spaces. The phase functions belong to a class of nondegenerate generalized quadratic forms that includes Schrödinger propagators and pseudodifferential operators. As a byproduct we obtain a characterization of all exponents $p, q, r_1, r_2, t_1, t_2 \in [1, \infty]$ of modulation spaces such that a symbol in $M^{p,q}(\mathbb{R}^{2d})$ gives a pseudodifferential operator that is continuous from $M^{r_1,r_2}(\mathbb{R}^d)$ into $M^{t_1,t_2}(\mathbb{R}^d)$.

1. INTRODUCTION

Fourier integral operators (FIOs) represent a mathematical tool to study the behavior of the solutions to partial differential equations. Our type of FIOs has its origins in Quantum Mechanics: they arise naturally in the study of the Cauchy problem for Schrödinger-type operators. We refer the reader to the pioneering works of Asada and Fujiwara [1], Cordoba and Fefferman [16], and Helffer and Robert [31]. This paper is concerned with the study of FIOs formally defined by

$$(1.1) \quad Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,\eta)} \sigma(x,\eta) \hat{f}(\eta) d\eta.$$

The functions σ and Φ are called symbol (or amplitude) and phase function, respectively. Our phase functions Φ , sometimes called “tame” [9, 10], are real-valued, smooth functions on \mathbb{R}^{2d} , satisfying $\partial_z^\alpha \Phi \in L^\infty(\mathbb{R}^{2d})$ for $\alpha \geq 2$, and the non-degeneracy condition

$$(1.2) \quad \left| \det \left(\frac{\partial^2 \Phi}{\partial x_i \partial \eta_l} \Big|_{(x,\eta)} \right) \right| \geq \delta > 0 \quad \forall (x,\eta) \in \mathbb{R}^{2d}.$$

Basic examples are provided by quadratic forms in the variables $x, \eta \in \mathbb{R}^d$ and the corresponding FIOs are the so called generalized metaplectic operators [10, 25]. Another well-known example is the phase

Date: July 16, 2012.

2010 *Mathematics Subject Classification.* Primary 35S30; Secondary 35S05, 42B35.

Key words and phrases. Schrödinger type propagators, pseudodifferential operators, modulation spaces.

$\Phi(x, \eta) = x \cdot \eta$ which gives pseudodifferential operators in the Kohn-Nirenberg form. Note that these phase functions differ from those of FIOs arising in the solutions of hyperbolic equations, that are positively homogeneous of degree one in η (see e.g. [15, 32, 37, 38]).

The aim of this paper is to provide optimal boundedness results for FIOs of the type above having rough symbols. The symbol classes that are suitable for this study reveal to be the so-called modulation spaces, introduced by Feichtinger in 1983 [20] and recalled in Subsection 2.1 below. Modulation spaces will be employed both for symbol spaces and spaces on which operators act.

Sharpness results in this framework were already pursued in the papers [13, 14, 39], where symbols in the particular modulation space $M^{\infty,1}(\mathbb{R}^{2d})$ were considered. Other results in this connection are contained in [6, 7, 42, 46].

The special case of pseudodifferential operators have been studied in the context of modulation spaces by several authors, including the earlier works by Gröchenig and Heil [28, 29], Labate [34, 35], Sjöstrand [39], Tachizawa [41]. Recent contributions are provided by [2, 3, 12, 17, 40, 44, 45]. For simplicity, let us first present our results in terms of pseudodifferential operators. The following sufficient conditions enlarge Toft's conditions [44, Theorem 4.3], whereas the necessary conditions contain those in [12, Proposition 5.3].

Theorem 1.1. *Assume that $1 \leq p, q, r_i, t_i \leq \infty$, $i = 1, 2$. Then the pseudodifferential operator T , from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, having symbol $\sigma \in M^{p,q}(\mathbb{R}^{2d})$, extends uniquely to a bounded operator from $\mathcal{M}^{r_1,r_2}(\mathbb{R}^d)$ to $\mathcal{M}^{t_1,t_2}(\mathbb{R}^d)$, with the estimate*

$$(1.3) \quad \|Tf\|_{\mathcal{M}^{t_1,t_2}} \lesssim \|\sigma\|_{M^{p,q}} \|f\|_{\mathcal{M}^{r_1,r_2}}$$

if and only if

$$(1.4) \quad 1/r_i - 1/t_i \geq 1 - 1/p - 1/q, \quad i = 1, 2,$$

and

$$(1.5) \quad q \leq \min(t_1, t_2, r'_1, r'_2).$$

This result can be seen as a characterization of pseudodifferential operators acting on modulation spaces, which completes the previous studies on this topic.

The sufficient conditions are obtained as a corollary of more general results for FIOs, contained in Theorem 3.9 below. Let us give an overview of our results in this framework.

Our main theme is to derive interpolation-theoretic consequences of the boundedness results for FIOs in [13, 14] and their possible sharpness. These prior results treat symbols in the modulation space $M^{\infty,1}(\mathbb{R}^{2d})$, possibly with a spatial weight or additional constraints on the phase function, and the continuity of the corresponding FIOs

acting either on $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$ or from $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$ to $\mathcal{M}^{r_2, r_1}(\mathbb{R}^d)$. We consider more general modulation spaces $M^{p, q}(\mathbb{R}^{2d})$ as symbol classes and studying the action from $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$ to $\mathcal{M}^{t_1, t_2}(\mathbb{R}^d)$, $1 \leq r_i, t_i \leq \infty$, $i = 1, 2$.

First we show that a symbol in $M^\infty(\mathbb{R}^{2d})$ gives rise to a FIO that maps $\mathcal{M}^1(\mathbb{R}^d)$ into $\mathcal{M}^\infty(\mathbb{R}^d)$ continuously, and a symbol in $M^1(\mathbb{R}^{2d})$ gives rise to a FIO that maps $\mathcal{M}^\infty(\mathbb{R}^d)$ into $\mathcal{M}^1(\mathbb{R}^d)$ continuously. These results are similar to results by Concetti, Garello and Toft [6, 7, 46].

Using complex interpolation and the results of [13] we then deduce continuity of FIOs with symbols in $M^{p, q}(\mathbb{R}^{2d})$ acting from $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$ to $\mathcal{M}^{t_1, t_2}(\mathbb{R}^d)$, and search for the weakest possible conditions on the family of exponents $p, q, r_1, r_2, t_1, t_2 \in [1, \infty]$ that admit continuity. If we make the additional assumption on the phase function

$$(1.6) \quad \sup_{x, x', \eta \in \mathbb{R}^d} |\nabla_x \Phi(x, \eta) - \nabla_x \Phi(x', \eta)| < \infty,$$

then the corresponding FIO T is continuous and satisfies (1.3) if and only if (1.4) and (1.5) hold, see Theorem 3.9 and Remark 3.10. Note that (1.6) is satisfied in the special case of $\Phi(x, \eta) = x \cdot \eta$, i.e. T is a pseudodifferential operator.

If we omit the assumption (1.6) and study the action on spaces \mathcal{M}^{r_1, r_2} , with $r_1 \neq r_2$, then the behavior of a FIO T is more troublesome. For instance, let us study the boundedness of T on \mathcal{M}^{r_1, r_2} , with $r_1 \neq r_2$. Consider the pointwise multiplication operator $Tf(x) = e^{\pi i |x|^2} f(x)$, which can be seen as a FIO with phase function $\Phi(x, \eta) = x \cdot \eta + |x|^2/2$ (that does not satisfy (1.6)), and symbol $\sigma \equiv 1 \in M^{\infty, 1}(\mathbb{R}^{2d})$. Taking $t_i = r_i$, $i = 1, 2$, the conditions (1.4) and (1.5) are satisfied but the operator T is bounded on \mathcal{M}^{r_1, r_2} if and only if $r_1 = r_2$, cf. [14, Proposition 7.1].

Nevertheless, if we do not assume (1.6) we can still obtain continuity on \mathcal{M}^{r_1, r_2} for all $r_1, r_2 \in [1, \infty]$, provided we introduce weights on the symbol spaces such that the symbols decay faster at infinity. Our main result in this direction is provided by Theorem 3.12 below.

Finally, motivated by the search for fixed-time estimates for one-parameter Schrödinger-type propagators (see [11, Section 4] and [13, Section 5]), we study in detail the action of a Fourier integral operator T from the spaces \mathcal{M}^{r_1, r_2} into \mathcal{M}^{r_2, r_1} , $r_2 \leq r_1$ (and analogously for Wiener amalgam spaces). We end by discussing the sharpness of the results. This topic is detailed in Section 3.2.

Notation. The Schwartz space is denoted by $\mathcal{S}(\mathbb{R}^d)$ and the tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ is normalized as $\mathcal{F}f(\eta) = \hat{f}(\eta) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \eta} dx$. For $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$ we set $v_s(x) = \langle x \rangle^s = (1 + |x|^2)^{s/2}$, and $x \cdot \eta$ denotes the inner product on \mathbb{R}^d . The notation $f \lesssim g$ means that there exists a positive

constant $C > 0$ such that $f \leq Cg$ (uniformly over all arguments of f and g where appropriate), while $f \asymp g$ means $f \lesssim g$ and $g \lesssim f$. Translations are denoted by $T_x f(y) = f(y - x)$ and modulations by $M_\eta f(x) = e^{2\pi i x \cdot \eta} f(x)$, $x, y, \eta \in \mathbb{R}^d$, $f \in \mathcal{S}(\mathbb{R}^d)$. The inner product on $L^2(\mathbb{R}^d)$ is conjugate linear in the second argument and is denoted by $\langle \cdot, \cdot \rangle$, which also denotes the conjugate linear action of \mathcal{S}' on \mathcal{S} .

2. PRELIMINARIES

In order to emphasize that T defined by (1.1) depends on σ we sometimes write $T = T_\sigma$.

Definition 2.1. *A real-valued phase functions Φ is called tame ([9, 13, 14]) provided the following three conditions are satisfied:*

- (i) $\Phi \in C^\infty(\mathbb{R}^{2d})$;
- (ii) there exist constants $C_\alpha > 0$ such that

$$(2.1) \quad |\partial^\alpha \Phi(x, \eta)| \leq C_\alpha \quad \forall \alpha \in \mathbb{N}^{2d}, \quad |\alpha| \geq 2;$$
- (iii) Φ satisfies the non-degeneracy condition (1.2).

2.1. Modulation spaces. [20, 21, 22, 23, 24, 27, 47]

In order to define modulation spaces we use the short-time Fourier transform (STFT) $V_g f$ of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to a nonzero window $g \in \mathcal{S}(\mathbb{R}^d)$. It is defined as $V_g f(x, \eta) = \mathcal{F}(f T_x \bar{g})(\eta)$ for $x, \eta \in \mathbb{R}^d$. For $f \in \mathcal{S}(\mathbb{R}^d)$ we have

$$(2.2) \quad V_g f(x, \eta) = \langle f, M_\eta T_x g \rangle = \int_{\mathbb{R}^d} e^{-2\pi i \eta \cdot y} f(y) \overline{g(y - x)} dy, \quad x, \eta \in \mathbb{R}^d.$$

The inversion formula for the STFT (see e.g. ([27, Corollary 3.2.3]) reads: If $\|g\|_{L^2} = 1$ and $f \in \mathcal{S}(\mathbb{R}^d)$ then

$$(2.3) \quad f = \int_{\mathbb{R}^{2d}} V_g f(x, \eta) M_\eta T_x g dx d\eta.$$

The following property of the STFT [27, Lemma 11.3.3] is useful when one needs to change window function.

Lemma 2.2. *If $f \in \mathcal{S}'(\mathbb{R}^d)$, $g_0, g_1, \gamma \in \mathcal{S}(\mathbb{R}^d)$ and $\langle \gamma, g_1 \rangle \neq 0$, then*

$$|V_{g_0} f(x, \eta)| \leq \frac{1}{|\langle \gamma, g_1 \rangle|} (|V_{g_1} f| * |V_{g_0} \gamma|)(x, \eta), \quad x, \eta \in \mathbb{R}^d.$$

In order to define the weighted modulation spaces of the symbols, we first introduce the class $\mathcal{M}_{v_s}(\mathbb{R}^{2d})$, $s \geq 0$, consisting of weights m that are positive measurable functions on \mathbb{R}^{2d} and satisfy $m(x + y) \lesssim v_s(x)m(y)$, $x, y \in \mathbb{R}^{2d}$. It follows that v_t is v_s -moderate for all $t \in \mathbb{R}$ such that $|t| \leq s$. In particular, we shall consider the class of weight functions on \mathbb{R}^{2d} given by $v_{s_1, s_2}(x, \eta) = \langle x \rangle^{s_1} \langle \eta \rangle^{s_2}$, $s_1, s_2 \in \mathbb{R}$, $x, \eta \in \mathbb{R}^d$, and $m = v_{s_1, s_2} \otimes 1$ on \mathbb{R}^{4d} .

Given a window function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, $m \in \mathcal{M}_{v_s}(\mathbb{R}^{2d})$ for some $s \geq 0$, and $1 \leq p, q \leq \infty$, the *modulation space* $M_m^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$ (weighted mixed-norm Lebesgue space). The norm on $M_m^{p,q}$ is

$$\|f\|_{M_m^{p,q}} = \|V_g f\|_{L_m^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \eta)|^p m(x, \eta)^p dx \right)^{q/p} d\eta \right)^{1/q}$$

(with natural modifications when $p = \infty$ or $q = \infty$). If $p = q$, we write M_m^p instead of $M_m^{p,p}$, and if $m \equiv 1$ on \mathbb{R}^{2d} , then we write $M^{p,q}$ and M^p for $M_m^{p,q}$ and $M_m^{p,p}$, respectively. The space $M_m^{p,q}(\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window g , in the sense that different nonzero window functions yield equivalent norms. The modulation space $M^{\infty,1}$ is also called Sjöstrand's class [39].

The closure of $\mathcal{S}(\mathbb{R}^d)$ in the $M_m^{p,q}$ -norm is denoted $\mathcal{M}_m^{p,q}(\mathbb{R}^d)$. Then $\mathcal{M}_m^{p,q}(\mathbb{R}^d) \subseteq M_m^{p,q}(\mathbb{R}^d)$, and $\mathcal{M}_m^{p,q}(\mathbb{R}^d) = M_m^{p,q}(\mathbb{R}^d)$ provided $p < \infty$ and $q < \infty$. For any $p, q \in [1, \infty]$ and any $m \in \mathcal{M}_{v_s}(\mathbb{R}^{2d})$, $s \geq 0$, we have: The inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ extends to a continuous sesquilinear map $M_m^{p,q}(\mathbb{R}^d) \times M_{1/m}^{p',q'}(\mathbb{R}^d) \rightarrow \mathbb{C}$. Here and elsewhere the conjugate exponent p' of $p \in [1, \infty]$ is defined by $1/p + 1/p' = 1$. If

$$\|f\| = \sup |\langle f, g \rangle|,$$

with supremum taken over all $g \in \mathcal{S}(\mathbb{R}^d)$ such that $\|g\|_{M_{1/m}^{p',q'}} \leq 1$, then $\|\cdot\|$ and $\|\cdot\|_{M_m^{p,q}}$ are equivalent norms (cf. [45, Proposition 1.4 (3)]). This result will be invoked using the phrase “by duality”.

Suppose $m_1, m_2 \in \mathcal{M}_{v_s}(\mathbb{R}^{2d})$ for some $s \geq 0$. Then we have the embeddings

$$(2.4) \quad \begin{aligned} \mathcal{S}(\mathbb{R}^d) &\subseteq M_{m_1}^{p_1,q_1}(\mathbb{R}^d) \subseteq M_{m_2}^{p_2,q_2}(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d), \\ p_1 &\leq p_2, \quad q_1 \leq q_2, \quad m_2 \lesssim m_1. \end{aligned}$$

Modulation spaces are closed under complex interpolation as follows (cf. [19]). If $p_j, q_j \in [1, \infty]$, $m_j \in \mathcal{M}_{v_s}(\mathbb{R}^{2d})$, $j = 1, 2$, for some $s \geq 0$, $0 < \theta < 1$,

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \quad m = m_1^{1-\theta} m_2^\theta,$$

then

$$(2.5) \quad (\mathcal{M}_{m_1}^{p_1,q_1}(\mathbb{R}^d), \mathcal{M}_{m_2}^{p_2,q_2}(\mathbb{R}^d))_{[\theta]} = \mathcal{M}_m^{p,q}(\mathbb{R}^d).$$

We need the following result concerning the modulation space norm of distributions of compact support in time or in frequency (cf., e.g., [6, 21, 36, 43]).

Lemma 2.3. *Let $1 \leq p, q \leq \infty$.*

(i) *For every $u \in \mathcal{S}'(\mathbb{R}^d)$, supported in a compact set $K \subseteq \mathbb{R}^d$, we have*

$u \in M^{p,q} \Leftrightarrow u \in \mathcal{FL}^q$, and

$$(2.6) \quad C_K^{-1} \|u\|_{M^{p,q}} \leq \|u\|_{\mathcal{FL}^q} \leq C_K \|u\|_{M^{p,q}},$$

where $C_K > 0$ depends only on K .

(ii) For every $u \in \mathcal{S}'(\mathbb{R}^d)$, whose Fourier transform is supported in a compact set $K \subseteq \mathbb{R}^d$, we have $u \in M^{p,q} \Leftrightarrow u \in L^p$, and

$$(2.7) \quad C_K^{-1} \|u\|_{M^{p,q}} \leq \|u\|_{L^p} \leq C_K \|u\|_{M^{p,q}},$$

where $C_K > 0$ depends only on K .

We refer to Gröchenig's book [27] for further properties of the modulation spaces.

Parseval's formula gives $|V_g f(x, \eta)| = |V_{\hat{g}} \hat{f}(\eta, -x)| = |\mathcal{F}(\hat{f} T_{\eta} \bar{\hat{g}})(-x)|$ for $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$. Hence

$$\|f\|_{M^{p,q}} = \left(\int_{\mathbb{R}^d} \|\hat{f} T_{\eta} \bar{\hat{g}}\|_{\mathcal{FL}^p}^q d\eta \right)^{1/q} = \|\hat{f}\|_{W(\mathcal{FL}^p, L^q)}.$$

Here $W(\mathcal{FL}^p, L^q)(\mathbb{R}^d)$ are particular cases of *Wiener amalgam spaces* with local component $\mathcal{FL}^p(\mathbb{R}^d)$ and global component $L^q(\mathbb{R}^d)$. It follows that we have $\mathcal{F}(M^{p,q}) = W(\mathcal{FL}^p, L^q)$. The closure of $\mathcal{S}(\mathbb{R}^d)$ in the $W(\mathcal{FL}^p, L^q)$ -norm is denoted $\mathcal{W}(\mathcal{FL}^p, L^q)$. For more information on Wiener amalgam spaces we refer to [18, 19, 26, 30].

We will need the modulation space norm of a complex Gaussian.

Lemma 2.4. *For $a > 0$, $b \in \mathbb{R}$, set $h_{a+ib}(x) = e^{-\pi(a+ib)|x|^2}$. Then we have for all $1 \leq p, q \leq \infty$*

$$(2.8) \quad \|h_{a+ib}\|_{M^{p,q}} \asymp \frac{((a+1)^2 + b^2)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{2})}}{a^{\frac{d}{2q}} (a(a+1) + b^2)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}}.$$

Proof. For $G_{a+ib}(x) = (a+ib)^{-d/2} e^{-\frac{\pi|x|^2}{a+ib}}$ we have by [13, Lemma 2.9]

$$\|G_{a+ib}\|_{W(\mathcal{FL}^p, L^q)} \asymp \frac{((a+1)^2 + b^2)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{2})}}{a^{\frac{d}{2q}} (a(a+1) + b^2)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}},$$

for all $1 \leq p, q \leq \infty$. Thus we obtain from $\mathcal{F}(M^{p,q}) = W(\mathcal{FL}^p, L^q)$

$$\|h_{a+ib}\|_{M^{p,q}} = \|\widehat{h_{a+ib}}\|_{W(\mathcal{FL}^p, L^q)} = \|G_{a+ib}\|_{W(\mathcal{FL}^p, L^q)}.$$

□

In particular we recover Toft's result [44, Lemma 1.8]. If $\varphi(x) = e^{-\pi|x|^2}$ and $\varphi_\lambda(x) = \varphi(\lambda x)$ then

$$(2.9) \quad \|\varphi_\lambda\|_{M^{p,q}} \asymp \lambda^{-d/p} (1 + \lambda)^{d(1/p+1/q-1)}, \quad \lambda > 0.$$

2.2. Continuity of Fourier integral operators on modulation spaces. Here we recollect and add comments on the results on Fourier integral operators and modulation spaces upon which the results in this paper build.

Assume that the phase function Φ is tame and satisfies the condition (1.6). Then a symbol that belongs to Sjöstrand's class $M^{\infty,1}(\mathbb{R}^{2d})$ gives rise to an operator that is continuous on $\mathcal{M}^{r_1,r_2}(\mathbb{R}^d)$, for every $1 \leq r_1, r_2 \leq \infty$. More precisely, the following [13, Theorem 1.1] holds.

Theorem 2.5. *Consider a tame phase function Φ satisfying (1.6), and a symbol $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$. Then the corresponding FIO T extends to a bounded operator on $\mathcal{M}^{r_1,r_2}(\mathbb{R}^d)$, for every $1 \leq r_1, r_2 \leq \infty$, with the estimate*

$$\|Tf\|_{\mathcal{M}^{r_1,r_2}} \lesssim \|\sigma\|_{M^{\infty,1}} \|f\|_{\mathcal{M}^{r_1,r_2}}.$$

Note that the pseudodifferential operator phase function $\Phi(x, \eta) = x \cdot \eta$ satisfies the assumptions of Theorem 2.5. If we omit the assumption (1.6) we can still get continuity on \mathcal{M}^{r_1,r_2} for all $r_1, r_2 \in [1, \infty]$, provided we introduce weights on the symbols according to the following result [13, Theorem 1.2].

Theorem 2.6. *Consider a tame phase function Φ , a symbol $\sigma \in M_{v_{s_1,s_2} \otimes 1}^{\infty,1}(\mathbb{R}^{2d})$, $s_1, s_2 \in \mathbb{R}$ and $1 \leq r_1, r_2 \leq \infty$. Assume one of the following conditions:*

- (i) $r_1 = r_2$ and $s_1, s_2 \geq 0$,
- (ii) $r_2 < r_1$, $s_1 > d \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$ and $s_2 \geq 0$,
- (iii) $r_1 < r_2$, $s_1 \geq 0$ and $s_2 > d \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$.

Then the corresponding FIO T extends to a bounded operator on $\mathcal{M}^{r_1,r_2}(\mathbb{R}^d)$, and

$$\|Tf\|_{\mathcal{M}^{r_1,r_2}} \lesssim \|\sigma\|_{M_{v_{s_1,s_2} \otimes 1}^{\infty,1}} \|f\|_{\mathcal{M}^{r_1,r_2}}.$$

The next quoted result [13, Theorem 1.3] shows, in particular, that a further condition on the phase function (2.10) gives a FIO that is continuous from $\mathcal{M}^{r_1,r_2}(\mathbb{R}^d)$ into $\mathcal{M}^{r_2,r_1}(\mathbb{R}^d)$.

Theorem 2.7. *Consider a tame phase function Φ , and let $1 \leq r_2 \leq r_1 \leq \infty$. Assume one of the following conditions:*

- (i) $s_1, s_2 \geq 0$, the symbol $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ and for some $\delta > 0$,

$$(2.10) \quad \left| \det \left(\frac{\partial^2 \Phi}{\partial x_i \partial x_l} \Big|_{(x,\eta)} \right) \right| \geq \delta \quad \forall (x, \eta) \in \mathbb{R}^{2d},$$

- (ii) the symbol $\sigma \in M_{v_{s_1,s_2} \otimes 1}^{\infty,1}(\mathbb{R}^{2d})$, with $s_1 > d \left(\frac{1}{r_2} - \frac{1}{r_1} \right)$ and $s_2 \geq 0$.

Then the corresponding FIO T extends to a bounded operator from $\mathcal{M}^{r_1,r_2}(\mathbb{R}^d)$ into $\mathcal{M}^{r_2,r_1}(\mathbb{R}^d)$, and

$$\|Tf\|_{\mathcal{M}^{r_2,r_1}} \lesssim \|\sigma\|_{M_{v_{s_1,s_2} \otimes 1}^{\infty,1}} \|f\|_{\mathcal{M}^{r_1,r_2}}.$$

A typical tame phase function that satisfies (2.10) is $\Phi(x, \eta) = x \cdot \eta + |x|^2/2$. When the symbol is $\sigma \equiv 1 \in M^{\infty,1}$, the FIO is the pointwise multiplication operator

$$(2.11) \quad Tf(x) = e^{i\pi|x|^2} f(x), \quad x \in \mathbb{R}^d.$$

By Theorem 2.7 and [13, Proposition 6.6], continuity from $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$ into $\mathcal{M}^{r_2, r_1}(\mathbb{R}^d)$ holds for this operator if and only if $r_2 \leq r_1$. Continuity of the operator (2.11) from $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$ into $\mathcal{M}^{t_1, t_2}(\mathbb{R}^d)$ is equivalent to continuity of the Schrödinger multiplier operator

$$(2.12) \quad Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \eta} e^{\pi i |\eta|^2} \hat{f}(\eta) d\eta, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

from $\mathcal{W}(\mathcal{FL}^{r_1}, L^{r_2})$ into $\mathcal{W}(\mathcal{FL}^{t_1}, L^{t_2})$.

For $1 \leq r_1 < r_2 \leq \infty$, one could conjecture that a FIO T_σ is bounded from \mathcal{M}^{r_1, r_2} to \mathcal{M}^{r_2, r_1} , provided $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$, and the phase function Φ is tame and satisfies

$$(2.13) \quad \left| \det \left(\frac{\partial^2 \Phi}{\partial \eta_i \partial \eta_l} \Big|_{(x, \eta)} \right) \right| \geq \delta \quad \forall (x, \eta) \in \mathbb{R}^{2d},$$

for some $\delta > 0$, instead of (2.10). But the conjecture is false as shown by the following result. It treats the Schrödinger multiplier (2.12), which is a FIO with phase $\Phi(x, \eta) = x \cdot \eta + |\eta|^2/2$ and symbol $\sigma \equiv 1 \in M^{\infty,1}$.

Proposition 2.8. *The Schrödinger multiplier (2.12) is bounded from $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$ to $\mathcal{M}^{t_1, t_2}(\mathbb{R}^d)$ if and only if $r_i \leq t_i$, $i = 1, 2$.*

Proof. The sufficiency of the condition follows immediately by combining the boundedness result on $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$ provided by [2, Theorem 1], and the inclusion relations for modulation spaces (2.4). Vice versa, assume that

$$\|Tf\|_{\mathcal{M}^{t_1, t_2}(\mathbb{R}^d)} \lesssim \|f\|_{\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)}, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Taking $f = \varphi_\lambda(t) = e^{-\pi\lambda^2|t|^2}$, (2.9) gives

$$(2.14) \quad \|\varphi_\lambda\|_{\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)} \asymp \begin{cases} \lambda^{-\frac{d}{r_1}}, & \lambda \rightarrow 0 \\ \lambda^{-\frac{d}{r_2}}, & \lambda \rightarrow +\infty. \end{cases}$$

Straightforward computations give

$$T\varphi_\lambda = (1 - i\lambda^2)^{-d/2} e^{-\pi \frac{\lambda^2}{1 - i\lambda^2} |\cdot|^2},$$

and an application of Lemma 2.4 yields

$$\|T\varphi_\lambda\|_{\mathcal{M}^{t_1, t_2}} \asymp \begin{cases} \lambda^{-\frac{d}{t_1}}, & \lambda \rightarrow 0 \\ \lambda^{-\frac{d}{t_2}}, & \lambda \rightarrow +\infty. \end{cases}$$

Combining with (2.14) we obtain $r_1 \leq t_1$ for $\lambda \rightarrow 0$, and $r_2 \leq t_2$ for $\lambda \rightarrow +\infty$. \square

2.3. Fourier integral operators and Wiener amalgam spaces.

From Theorems 2.5, 2.6 and 2.7 we may infer results on continuity of FIOs acting on Wiener amalgam spaces. Indeed, since $\mathcal{FM}^{r_1, r_2} = \mathcal{W}(\mathcal{FL}^{r_1}, L^{r_2})$, it follows that

$$(2.15) \quad \|Tf\|_{\mathcal{W}(\mathcal{FL}^{t_1}, L^{t_2})} \lesssim \|\sigma\|_{M_{v_{s_1}, s_2 \otimes 1}^{p, q}} \|f\|_{\mathcal{W}(\mathcal{FL}^{r_1}, L^{r_2})}$$

if and only if

$$\|\tilde{T}f\|_{\mathcal{M}^{t_1, t_2}} \lesssim \|\sigma\|_{M_{v_{s_1}, s_2 \otimes 1}^{p, q}} \|f\|_{\mathcal{M}^{r_1, r_2}},$$

where $\tilde{T} = \mathcal{F} \circ T \circ \mathcal{F}^{-1}$. By duality and an explicit computation this is equivalent to verifying that the adjoint operator

$$(2.16) \quad \tilde{T}^* f(x) = \int e^{-2\pi i \Phi(-\eta, x)} \overline{\sigma(-\eta, x)} \hat{f}(\eta) d\eta$$

extends to a bounded operator from $\mathcal{M}^{t'_1, t'_2}(\mathbb{R}^d)$ to $\mathcal{M}^{r'_1, r'_2}(\mathbb{R}^d)$. Since $\sigma \in M_{v_{s_1}, s_2 \otimes 1}^{p, q}(\mathbb{R}^{2d})$ if and only if $\tilde{\sigma}(x, \eta) = \overline{\sigma(-\eta, x)} \in M_{v_{s_2}, s_1 \otimes 1}^{p, q}(\mathbb{R}^{2d})$ (cf. [13, Lemma 2.10]), the continuity statement for Wiener amalgam spaces in (2.15) follows from the continuity estimate

$$(2.17) \quad \|\tilde{T}^* f\|_{\mathcal{M}^{r'_1, r'_2}} \lesssim \|\tilde{\sigma}\|_{M_{v_{s_2}, s_1 \otimes 1}^{p, q}} \|f\|_{\mathcal{M}^{t'_1, t'_2}}.$$

These considerations immediately transfer continuity results for FIOs acting on modulation spaces to FIOs acting on Wiener amalgam spaces, possibly with modified assumptions on the phase function, cf. [13, Corollary 3.9 and Corollary 5.2]. We shall state continuity results for FIOs acting on Wiener amalgam spaces as corollaries of the corresponding continuity results for modulation spaces.

2.4. A characterization of modulation spaces. We recall the formula, obtained in [14, Section 6] (see also [13, Proposition 3.2] and [10, Section 4]), which expresses the Gabor matrix of the FIO T in terms of the STFT of its symbol σ . Suppose the phase function Φ satisfy (i) and (ii) of Definition 2.1. Choose a non-zero window function $g \in \mathcal{S}(\mathbb{R}^d)$, and define for $z, \zeta \in \mathbb{R}^{2d}$

$$(2.18) \quad \Psi_z(\zeta) := e^{2\pi i \Phi_{2, z}(\zeta)} (\bar{g} \otimes \hat{g})(\zeta),$$

where

$$(2.19) \quad \Phi_{2, z}(\zeta) = 2 \sum_{|\alpha|=2} \int_0^1 (1-t) \partial^\alpha \Phi(z + t\zeta) dt \frac{\zeta^\alpha}{\alpha!}, \quad z, \zeta \in \mathbb{R}^{2d}.$$

Let $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ and denote $g_{x, \eta} = M_\eta T_x g$ for $x, \eta \in \mathbb{R}^d$. Then the so called Gabor matrix of T , given by $\langle Tg_{x, \eta}, g_{x', \eta'} \rangle$, can be expressed via the STFT as

$$(2.20) \quad |\langle Tg_{x, \eta}, g_{x', \eta'} \rangle| = |V_{\Psi_{(x', \eta')}} \sigma(x', \eta, \eta' - \nabla_x \Phi(x', \eta), x - \nabla_\eta \Phi(x', \eta))|,$$

for $x, \eta, x', \eta' \in \mathbb{R}^d$.

We present a characterization of the spaces $M_{1 \otimes m}^{p,q}(\mathbb{R}^{2d})$ when $m \in \mathcal{M}_{v_s}(\mathbb{R}^{2d})$ for $s \geq 0$. This is a generalization of [9, Proposition 3.10], that treats the cases $(p, q) = (\infty, 1)$ and $p = q = \infty$, to general $p, q \in [1, \infty]$. This characterization shows that the time-frequency concentration of the symbol σ does not depend on the parameter $z \in \mathbb{R}^{2d}$ of the window Ψ_z , defined in (2.18).

First, we need the following simplified version of [9, Lemma 3.9].

Lemma 2.9. *If $s \geq 0$, $g \in \mathcal{S}(\mathbb{R}^d)$ and $\Psi \in \mathcal{S}(\mathbb{R}^{2d})$ then*

$$(2.21) \quad \sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \Psi| \in L_{1 \otimes v_s}^1(\mathbb{R}^{4d}).$$

The characterization for modulation spaces in terms of Ψ_z is as follows.

Proposition 2.10. *Let $s \geq 0$, $m \in \mathcal{M}_{v_s}(\mathbb{R}^{2d})$, $p, q \in [1, \infty]$, and $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$. Then*

$$\sigma \in M_{1 \otimes m}^{p,q}(\mathbb{R}^{2d}) \iff \sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \sigma| \in L_{1 \otimes m}^{p,q}(\mathbb{R}^{4d}),$$

and

$$(2.22)$$

$$\begin{aligned} \|\sigma\|_{M_{1 \otimes m}^{p,q}(\mathbb{R}^{2d})} &\asymp \left\| \sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \sigma| \right\|_{L_{1 \otimes m}^{p,q}(\mathbb{R}^{4d})} \\ &= \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} \sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \sigma(u_1, u_2)|^p m(u_2)^p du_1 \right)^{\frac{q}{p}} du_2 \right)^{\frac{1}{q}} \end{aligned}$$

(with obvious modifications when $p = \infty$ or $q = \infty$).

Proof. We first prove $\|\sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \sigma|\|_{L_{1 \otimes m}^{p,q}} \lesssim \|\sigma\|_{M_{1 \otimes m}^{p,q}}$. Taking $\Psi \in \mathcal{S}(\mathbb{R}^{2d})$ such that $\|\Psi\|_{L^2} = 1$ and using Lemma 2.2, we have

$$|V_{\Psi_z} \sigma|(u_1, u_2) \leq |V_{\Psi} \sigma| * |V_{\Psi_z} \Psi|(u_1, u_2) \leq |V_{\Psi} \sigma| * \sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \Psi|(u_1, u_2).$$

Young's inequality and the assumption $m \in \mathcal{M}_{v_s}(\mathbb{R}^{2d})$ yield

$$\left\| \sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \sigma| \right\|_{L_{1 \otimes m}^{p,q}} \leq \|V_{\Psi} \sigma\|_{L_{1 \otimes m}^{p,q}} \left\| \sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \Psi| \right\|_{L_{1 \otimes v_s}^1} \lesssim \|\sigma\|_{M_{1 \otimes m}^{p,q}},$$

thanks to Lemma 2.9.

On the other hand, assume $\|\sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \sigma|\|_{L_{1 \otimes m}^{p,q}(\mathbb{R}^{4d})} < \infty$. Since $\|\Psi_z\|_{L^2} = \|g\|_{L^2}^4$ and $|V_{\Psi} \Psi_z(u)| = |V_{\Psi_z} \Psi(-u)|$, denoting $\tilde{f}(x) = f(-x)$, Lemma 2.2 gives

$$\begin{aligned} |V_{\Psi} \sigma(u_1, u_2)| &\lesssim |V_{\Psi_z} \sigma| * |V_{\Psi_z} \Psi|(u_1, u_2) \\ &= |V_{\Psi_z} \sigma| * |\widetilde{V_{\Psi_z} \Psi}|(u_1, u_2) \\ &\leq \left(\sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \sigma| \right) * \left(\sup_{z \in \mathbb{R}^{2d}} |\widetilde{V_{\Psi_z} \Psi}| \right)(u_1, u_2). \end{aligned}$$

Applying Young's inequality and Lemma 2.9, we finally obtain

$$\begin{aligned} \|\sigma\|_{M_{1 \otimes m}^{p,q}} &= \|V_\Psi \sigma\|_{L_{1 \otimes m}^{p,q}} \lesssim \left\| \sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \sigma| \right\|_{L_{1 \otimes m}^{p,q}} \left\| \sup_{z \in \mathbb{R}^{2d}} |\widetilde{V_{\Psi_z} \Psi}| \right\|_{L_{1 \otimes v_s}^1} \\ &\lesssim \left\| \sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \sigma| \right\|_{L_{1 \otimes m}^{p,q}}. \end{aligned}$$

□

3. CONTINUITY RESULTS FOR FIOS

First we will prove two results concerning continuity of FIOs with symbols in $M^\infty(\mathbb{R}^{2d})$ and $M^1(\mathbb{R}^{2d})$, respectively. Then we will make complex interpolation between them and the results in Section 2.

We need the following Schur-type test, whose proof is obvious.

Lemma 3.1. *Consider an integral operator A on \mathbb{R}^{2d} , given by*

$$(Af)(x', \eta') = \iint_{\mathbb{R}^{2d}} K(x', \eta'; x, \eta) f(x, \eta) dx d\eta.$$

- (i) *If $K \in L^\infty(\mathbb{R}^{4d})$ then A is continuous from $L^1(\mathbb{R}^{2d})$ into $L^\infty(\mathbb{R}^{2d})$.*
- (ii) *If $K \in L^1(\mathbb{R}^{4d})$ then A is continuous from $L^\infty(\mathbb{R}^{2d})$ into $L^1(\mathbb{R}^{2d})$.*

Proposition 3.2. *Consider a tame phase function Φ and suppose $\sigma \in M^\infty(\mathbb{R}^{2d})$. Then T_σ extends to a bounded operator from $\mathcal{M}^1(\mathbb{R}^d)$ to $\mathcal{M}^\infty(\mathbb{R}^d)$, and*

$$(3.1) \quad \|T_\sigma f\|_{\mathcal{M}^\infty(\mathbb{R}^d)} \lesssim \|\sigma\|_{M^\infty(\mathbb{R}^{2d})} \|f\|_{\mathcal{M}^1(\mathbb{R}^d)}.$$

Proof. Let $g \in \mathcal{S}(\mathbb{R}^d)$ with $\|g\|_{L^2} = 1$. For $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the map $f \rightarrow \langle Tf, \varphi \rangle$, denoted u_φ , belongs to $\mathcal{S}'(\mathbb{R}^d)$. Since u_φ is linear (rather than antilinear) we obtain from the inversion formula (2.3) and [33, Theorem 5.1.1]

$$\begin{aligned} \langle Tf, \varphi \rangle &= \langle u_\varphi, \overline{f} \rangle = \langle u_\varphi, \int_{\mathbb{R}^{2d}} \overline{V_g f(x, \eta)} M_\eta T_x g(\cdot) dx d\eta \rangle \\ &= \int_{\mathbb{R}^{2d}} \langle u_\varphi, \overline{M_\eta T_x g} \rangle V_g f(x, \eta) dx d\eta \\ &= \int_{\mathbb{R}^{2d}} \langle T M_\eta T_x g, \varphi \rangle V_g f(x, \eta) dx d\eta. \end{aligned}$$

It follows that, for $f \in \mathcal{S}(\mathbb{R}^d)$,

$$V_g(Tf)(x', \eta') = \int_{\mathbb{R}^{2d}} \langle T g_{x, \eta}, g_{x', \eta'} \rangle V_g f(x, \eta) dx d\eta.$$

The desired estimate (3.1) thus follows if we can prove that the map K_T defined by

$$K_T G(x', \eta') = \int_{\mathbb{R}^{2d}} \langle T g_{x, \eta}, g_{x', \eta'} \rangle G(x, \eta) dx d\eta$$

is continuous from $L^1(\mathbb{R}^{2d})$ into $L^\infty(\mathbb{R}^{2d})$. By Lemma 3.1 (i) it suffices to prove that its integral kernel

$$K_T(x', \eta'; x, \eta) = \langle Tg_{x,\eta}, g_{x',\eta'} \rangle$$

satisfies $K_T \in L^\infty(\mathbb{R}^{4d})$. By (2.20) we have

$$\begin{aligned} |K_T(x', \eta'; x, \eta)| &= |V_{\Psi_{(x', \eta)}} \sigma(x', \eta, \eta' - \nabla_x \Phi(x', \eta), x - \nabla_\eta \Phi(x', \eta))| \\ &\leq \sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \sigma(x', \eta, \eta' - \nabla_x \Phi(x', \eta), x - \nabla_\eta \Phi(x', \eta))| \end{aligned}$$

and hence

$$\begin{aligned} &\sup_{(x, \eta, x', \eta') \in \mathbb{R}^{4d}} |K_T(x', \eta'; x, \eta)| \\ &\leq \sup_{(x, \eta, x', \eta') \in \mathbb{R}^{4d}} \sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \sigma(x', \eta, \eta' - \nabla_x \Phi(x', \eta), x - \nabla_\eta \Phi(x', \eta))| \\ &= \sup_{(x, \eta, x', \eta') \in \mathbb{R}^{4d}} \sup_{z \in \mathbb{R}^{2d}} |V_{\Psi_z} \sigma(x', \eta, \eta', x)| \asymp \|\sigma\|_{M^\infty} \end{aligned}$$

by the characterization (2.22). \square

Proceeding similarly as in the proof of Proposition 3.2, using Lemma 3.1 (ii) instead of (i), gives the following dual result.

Proposition 3.3. *Consider a tame phase function Φ and suppose $\sigma \in M^1(\mathbb{R}^{2d})$. Then T_σ extends to a bounded operator from $\mathcal{M}^\infty(\mathbb{R}^d)$ to $\mathcal{M}^1(\mathbb{R}^d)$, and*

$$(3.2) \quad \|T_\sigma f\|_{\mathcal{M}^1(\mathbb{R}^d)} \lesssim \|\sigma\|_{M^1(\mathbb{R}^{2d})} \|f\|_{\mathcal{M}^\infty(\mathbb{R}^d)}.$$

Remark 3.4. *We notice that Proposition 3.2 and Proposition 3.3 hold with weaker assumptions on the real-valued phase function. In fact, conditions (i), (ii) and (iii) of Definition 2.1 may evidently be relaxed to $\Phi \in C^\infty(\mathbb{R}^{2d})$ and $\sup_{|\alpha|=2} |\partial^\alpha \Phi| \lesssim v_N$ for some $N > 0$. A similar result, with assumptions on Φ that are weaker than (i), (ii) and (iii) but stronger than $\Phi \in C^2(\mathbb{R}^{2d})$ and $\sup_{|\alpha|=2} |\partial^\alpha \Phi| \lesssim v_N$, is shown in [7, Theorem 2.7]. More precisely, [7, Theorem 2.7] treats a more general type of FIO whose phase function depends on three variables as*

$$Tf(x) = \iint_{\mathbb{R}^{2d}} e^{2\pi i \varphi(x, y, \xi)} \sigma(x, \xi) f(y) dy d\xi.$$

Specializing to our situation, we have $\varphi(x, y, \xi) = \Phi(x, \xi) - y \cdot \xi$, and the sufficient condition on Φ in [7, Theorem 2.7] is $\partial^\alpha \Phi \in M^{\infty, 1}$ for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = 2$.

There is also a version of this result for weighted modulation spaces in [46, Proposition 3.1 (3)]. The symbol space, as well as the spaces between which the operator acts, are then weighted modulation spaces, with polynomially bounded weights that are related as described in [46, Proposition 3.1 (3)].

3.1. Results based on Theorems 2.5 and 2.6. Propositions 3.2 and 3.3 admit us to prove the following interpolation-theoretic consequences of Theorems 2.5 and 2.6. First we discuss the case when both the domain and the range are equal-index modulation spaces.

Theorem 3.5. *Consider a tame phase function Φ , and let $1 \leq p, q, r, t \leq \infty$. If*

$$(3.3) \quad q \leq \min(t, r') \quad \text{and} \quad 1/r - 1/t \geq 1 - 1/p - 1/q,$$

and $\sigma \in M^{p,q}(\mathbb{R}^{2d})$, then T extends to a bounded operator from $\mathcal{M}^r(\mathbb{R}^d)$ to $\mathcal{M}^t(\mathbb{R}^d)$, with

$$(3.4) \quad \|Tf\|_{\mathcal{M}^t(\mathbb{R}^d)} \lesssim \|\sigma\|_{M^{p,q}(\mathbb{R}^{2d})} \|f\|_{\mathcal{M}^r(\mathbb{R}^d)}.$$

Proof. If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ then T extends, according to Theorem 2.6 (i), to a bounded operator on $\mathcal{M}^s(\mathbb{R}^d)$ for all $1 \leq s \leq \infty$ and

$$(3.5) \quad \|T_\sigma f\|_{\mathcal{M}^s(\mathbb{R}^d)} \lesssim \|\sigma\|_{M^{\infty,1}(\mathbb{R}^{2d})} \|f\|_{\mathcal{M}^s(\mathbb{R}^d)}.$$

Regarding T as the bilinear map $(\sigma, f) \mapsto Tf$, (3.5) and (3.1) of Proposition 3.2 says that T is continuous

$$\begin{aligned} M^{\infty,1}(\mathbb{R}^{2d}) \times \mathcal{M}^s(\mathbb{R}^d) &\rightarrow \mathcal{M}^s(\mathbb{R}^d) \quad \text{for } 1 \leq s \leq \infty, \quad \text{and} \\ M^\infty(\mathbb{R}^{2d}) \times \mathcal{M}^1(\mathbb{R}^d) &\rightarrow \mathcal{M}^\infty(\mathbb{R}^d) \end{aligned}$$

Using multi-linear complex interpolation (cf. [4, Theorem 4.4.1]) and (2.5), it follows that the bilinear map T is continuous

$$(3.6) \quad T : \mathcal{M}^{\infty,q}(\mathbb{R}^{2d}) \times \mathcal{M}^r(\mathbb{R}^d) \rightarrow \mathcal{M}^t(\mathbb{R}^d),$$

for $q, r, t \in [1, \infty]$ such that $1/r - 1/t = 1 - 1/q$, $q \leq \min(t, r')$ and $r \leq t$. Likewise, interpolation between (3.6) and (3.2) of Proposition 3.3 gives (3.4) for $p, q, r, t \in [1, \infty]$ such that

$$q \leq \min(p, t, r') \quad \text{and} \quad 1/r - 1/t = 1 - 1/p - 1/q.$$

Due to the embeddings (2.4), we may relax these assumptions on r and t (possibly decreasing r and increasing t), keeping p, q fixed, into

$$(3.7) \quad q \leq \min(p, t, r') \quad \text{and} \quad 1/r - 1/t \geq 1 - 1/p - 1/q,$$

and (3.4) still holds true. Finally, again using the embeddings (2.4) in order to relax the conditions on p and q , possibly decreasing p and q while keeping r, t fixed, it can be verified that the result extends to all $p, q, r, t \in [1, \infty]$ such that

$$q \leq \min(t, r'), \quad 1/r - 1/t \geq 1 - 1/p - 1/q.$$

□

Remark 3.6. *For $p = \infty$ and $1 < q < \infty$, the sufficient condition on the symbol in Theorem 3.5 should be $\sigma \in \mathcal{M}^{\infty,q}(\mathbb{R}^{2d})$ rather than $\sigma \in M^{\infty,q}(\mathbb{R}^{2d})$. This small modification is understood in all results of this paper, but not spelled out in order not to burden the presentation.*

Remark 3.7. *The sufficient conditions in Theorem 3.9 are also necessary. Indeed, choose $\Phi(x, \eta) = x \cdot \eta$ which is tame. Then the corresponding FIO T reduces to a pseudodifferential operator and the necessary conditions are provided by Theorem 1.1, written for the case $r_1 = r_2 = r$, $t_1 = t_2 = t$.*

Remark 3.8. *We notice that a related result for weighted modulation spaces follows from a combination of [46, Proposition 1.10] and [46, Theorem 2.10]. In particular, it follows that a version of the inequality (3.5) holds for weighted spaces and symbols, for certain combinations of weights, and $1 < s < \infty$, when the phase function Φ is tame. However, we inform the reader that the condition on the phase function*

$$\left| \det \left(\frac{\partial^2 \varphi}{\partial y_i \partial \xi_l} \Big|_{(x, y, \xi)} \right) \right| \geq \delta \quad \forall (x, y, \xi) \in \mathbb{R}^{3d}$$

for some $\delta > 0$, specified as sufficient for the conclusions in [46, Proposition 3.1], is correct for parts (1), (2) and (3) of that proposition, but not for part (4).

Next we treat FIOs acting on modulation spaces from \mathcal{M}^{r_1, r_2} to \mathcal{M}^{t_1, t_2} with possibly $r_1 \neq r_2$ or $t_1 \neq t_2$. In this case the assumptions of Theorem 3.5 are not enough to provide boundedness (see the counterexample in the Introduction). Instead we obtain results by strengthening either the phase (Theorem 3.9) or the symbol (Theorem 3.12) hypotheses.

Since the arguments of the proof of the result below follow closely the proof of Theorem 3.5, starting from Theorem 2.5 and using Propositions 3.2 and 3.3, we omit its proof.

Theorem 3.9. *Let $1 \leq p, q, r_1, r_2, t_1, t_2 \leq \infty$, let the phase function Φ be tame and satisfy (1.6), and suppose (1.4) and (1.5) hold true. If $\sigma \in M^{p, q}(\mathbb{R}^{2d})$ then the corresponding operator T extends to a bounded operator from $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$ to $\mathcal{M}^{t_1, t_2}(\mathbb{R}^d)$, with*

$$(3.8) \quad \|Tf\|_{\mathcal{M}^{t_1, t_2}} \lesssim \|\sigma\|_{M^{p, q}} \|f\|_{\mathcal{M}^{r_1, r_2}}.$$

Remark 3.10. *The sufficient conditions in Theorem 3.9 are also necessary. Indeed, choose $\Phi(x, \eta) = x \cdot \eta$ which is tame and satisfies (1.6). Then the corresponding FIO T reduces to a pseudodifferential operator and the necessary conditions are provided by Theorem 1.1.*

As a consequence of Theorem 3.9 we obtain the following result for Wiener amalgam spaces.

Corollary 3.11. *Under the assumptions of Theorem 3.9, with (1.6) replaced by*

$$(3.9) \quad \sup_{x, \eta, \eta' \in \mathbb{R}^d} |\nabla_\eta \Phi(x, \eta) - \nabla_\eta \Phi(x, \eta')| < \infty,$$

the corresponding operator T extends to a bounded operator from the space $\mathcal{W}(\mathcal{FL}^{r_1}, L^{r_2})(\mathbb{R}^d)$ to $\mathcal{W}(\mathcal{FL}^{t_1}, L^{t_2})(\mathbb{R}^d)$, with

$$(3.10) \quad \|Tf\|_{\mathcal{W}(\mathcal{FL}^{t_1}, L^{t_2})(\mathbb{R}^d)} \lesssim \|\sigma\|_{M^{p,q}} \|f\|_{\mathcal{W}(\mathcal{FL}^{r_1}, L^{r_2})}.$$

If we do not assume the condition (1.6) on the phase, then similar FIO boundedness results can still be obtained by asking for more decay at infinity of the corresponding symbol. This means that we replace the unweighted modulation spaces $M^{p,q}$ by weighted spaces.

Theorem 3.12. *Let $1 \leq p, q, r_1, r_2, t_1, t_2 \leq \infty$ and suppose (1.4) holds. Consider a tame phase function Φ , and a symbol $\sigma \in M_{v_{s_1, s_2}^{p,q} \otimes 1}(\mathbb{R}^{2d})$, $s_1, s_2 \in \mathbb{R}$. Suppose furthermore that either of the following two requirements are satisfied.*

- (i) $s_2 \geq 0$, and either

$$q \leq \min(t_1, t_2, r'_1), \quad r_2 \leq r_1, \quad \text{and} \quad s_1 > d(1/r_2 - 1/r_1),$$
 or

$$q \leq \min(t_2, r'_1, r'_2), \quad t_2 \leq t_1, \quad \text{and} \quad s_1 > d(1/t_2 - 1/t_1).$$
- (ii) $s_1 \geq 0$, and either

$$q \leq \min(t_1, t_2, r'_2), \quad r_1 \leq r_2, \quad \text{and} \quad s_2 > d(1/r_1 - 1/r_2),$$
 or

$$q \leq \min(t_1, r'_1, r'_2), \quad t_1 \leq t_2, \quad \text{and} \quad s_2 > d(1/t_1 - 1/t_2).$$

Then T extends to a bounded operator from $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$ to $\mathcal{M}^{t_1, t_2}(\mathbb{R}^d)$, and

$$\|Tf\|_{\mathcal{M}^{t_1, t_2}} \lesssim \|\sigma\|_{M_{v_{s_1, s_2}^{p,q} \otimes 1}} \|f\|_{\mathcal{M}^{r_1, r_2}}.$$

Proof. The boundedness follows by complex interpolation, using Theorem 2.6, Propositions 3.2 and 3.3, as detailed below. By Theorem 2.6 (i) and (ii) we have

$$\|Tf\|_{\mathcal{M}^{r_1, r_2}} \lesssim \|\sigma\|_{M_{v_{s_1, 0}^{\infty, 1} \otimes 1}} \|f\|_{\mathcal{M}^{r_1, r_2}}$$

for $1 \leq r_2 \leq r_1 \leq \infty$ and $s_1 > d(1/r_2 - 1/r_1)$. Proposition 3.2 and interpolation give continuity of

$$(3.11) \quad T : \mathcal{M}_{v_{s, 0}^{\infty, q} \otimes 1}(\mathbb{R}^{2d}) \times \mathcal{M}^{r_1, r_2}(\mathbb{R}^d) \rightarrow \mathcal{M}^{t_1, t_2}(\mathbb{R}^d),$$

for $q, r_1, r_2, t_1, t_2 \in [1, \infty]$ and $s \in \mathbb{R}$ such that $1/r_i - 1/t_i = 1 - 1/q$ for $i = 1, 2$, $r_2 \leq r_1$, $q \leq \min(t_2, r'_1)$ and $s > d(1/r_2 - 1/r_1)$. Next interpolation between (3.11) and Proposition 3.3 gives

$$(3.12) \quad \|Tf\|_{\mathcal{M}^{t_1, t_2}} \lesssim \|\sigma\|_{\mathcal{M}_{v_{s, 0}^{p, q} \otimes 1}} \|f\|_{\mathcal{M}^{r_1, r_2}}$$

for $p, q, r_1, r_2, t_1, t_2 \in [1, \infty]$ and $s \in \mathbb{R}$ that satisfy

$$\begin{aligned} q &\leq \min(p, t_2, r'_1), \quad 1/r_i - 1/t_i = 1 - 1/p - 1/q \quad \text{for } i = 1, 2, \\ r_2 &\leq r_1, \quad \text{and } s > d(1/r_2 - 1/r_1). \end{aligned}$$

Invoking (2.4), we may relax the conditions on t_i, r_i for $i = 1, 2$. Thus we may possibly increase t_i and decrease r_i for $i = 1, 2$ while keeping p, q fixed, such that (1.4) holds, and either

$$q \leq \min(p, t_1, t_2, r'_1), \quad r_2 \leq r_1, \quad \text{and} \quad s > d(1/r_2 - 1/r_1),$$

or

$$q \leq \min(p, t_2, r'_1, r'_2), \quad t_2 \leq t_1, \quad \text{and} \quad s > d(1/t_2 - 1/t_1),$$

while preserving (3.12). Again using (2.4), these conditions may be further relaxed, in the sense of possibly decreasing p and q while keeping $r_i, t_i, i = 1, 2$, fixed, such that (1.4) holds, and either

$$q \leq \min(t_1, t_2, r'_1), \quad r_2 \leq r_1, \quad \text{and} \quad s > d(1/r_2 - 1/r_1),$$

or

$$q \leq \min(t_2, r'_1, r'_2), \quad t_2 \leq t_1, \quad \text{and} \quad s > d(1/t_2 - 1/t_1),$$

while maintaining (3.12). Finally, another appeal to (2.4) shows that $M_{v_{s_1, s_2} \otimes 1}^{p, q}(\mathbb{R}^{2d}) \subseteq M_{v_{s_1, 0} \otimes 1}^{p, q}(\mathbb{R}^{2d})$ which proves the Theorem under assumption (i).

If we instead use the assumption (ii), the theorem is proved with a similar argument, replacing Theorem 2.6 (ii) by Theorem 2.6 (iii) at the beginning. \square

Corollary 3.13. *Consider a phase Φ and a symbol σ satisfying the assumptions of Theorem 3.12. Then the corresponding operator T extends to a bounded operator from $\mathcal{W}(\mathcal{FL}^{r_1}, L^{r_2})(\mathbb{R}^d)$ to $\mathcal{W}(\mathcal{FL}^{t_1}, L^{t_2})(\mathbb{R}^d)$, with*

$$\|Tf\|_{\mathcal{W}(\mathcal{FL}^{t_1}, L^{t_2})(\mathbb{R}^d)} \lesssim \|\sigma\|_{M_{v_{s_1, s_2} \otimes 1}^{p, q}} \|f\|_{\mathcal{W}(\mathcal{FL}^{r_1}, L^{r_2})}.$$

3.2. Action from $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$ to $\mathcal{M}^{r_2, r_1}(\mathbb{R}^d)$. In this subsection we prove results for tame phase functions that satisfy (2.10). This setup is particularly useful to derive fixed-time estimates for a family of time-dependent FIOs $\{T_t\}_{t \in \mathbb{R}}$. These arise as solutions to Cauchy problems for partial differential equations. For instance, consider the propagators $T_t = e^{itH}$, where H is the Weyl quantization of a quadratic form on the phase space \mathbb{R}^{2d} . We refer e.g. to [11] and [13].

As a byproduct, we obtain continuity results for FIOs acting between Wiener amalgam spaces (see Corollary 3.15).

Theorem 3.14. *Consider a tame phase function Φ that satisfies (2.10), and $1 \leq p, q, r_1, r_2 \leq \infty$ such that*

$$(3.13) \quad r_2 \leq r_1, \quad q \leq \min(r_2, r'_1), \quad \frac{1}{p} + \frac{1}{q} \geq 1.$$

If the symbol $\sigma \in M^{p, q}(\mathbb{R}^{2d})$, then the corresponding FIO T extends to a bounded operator $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d) \rightarrow \mathcal{M}^{r_2, r_1}(\mathbb{R}^d)$, with

$$(3.14) \quad \|Tf\|_{\mathcal{M}^{r_2, r_1}} \lesssim \|\sigma\|_{M^{p, q}} \|f\|_{\mathcal{M}^{r_1, r_2}}.$$

Proof. First we observe that if $\sigma \in M^2(\mathbb{R}^{2d}) = L^2(\mathbb{R}^{2d})$ then $T = T_\sigma$ is bounded on $L^2(\mathbb{R}^d)$ with

$$(3.15) \quad \|T_\sigma f\|_{L^2(\mathbb{R}^d)} \leq \|\sigma\|_{L^2(\mathbb{R}^{2d})} \|f\|_{L^2(\mathbb{R}^d)}, \quad \forall f \in L^2(\mathbb{R}^d).$$

Indeed, using the Cauchy–Schwarz inequality and the Plancherel theorem, for every $f, g \in L^2(\mathbb{R}^d)$,

$$|\langle T_\sigma f, g \rangle| = \left| \langle e^{2\pi i \Phi} \sigma, \tilde{f} \otimes g \rangle \right| \leq \|\sigma\|_{L^2(\mathbb{R}^{2d})} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}$$

and (3.15) follows. Next, multilinear complex interpolation between Theorem 2.7 (i) and (3.15) yields the estimate (3.14), for $r_2 \leq r_1$, $q \leq \min(r_2, r'_1)$, $p \geq 2$ and $1/p + 1/q = 1$. Finally, the inclusion relations for modulation spaces (2.4) extend the result to $1 \leq p \leq \infty$ and $1/p + 1/q \geq 1$ (note that $q \leq 2$). \square

Corollary 3.15. *Under the assumptions of Theorem 3.14, with (2.10) replaced by (2.13), the operator T extends to a bounded operator from $\mathcal{W}(\mathcal{F}L^{r_1}, L^{r_2})(\mathbb{R}^d)$ to $\mathcal{W}(\mathcal{F}L^{r_2}, L^{r_1})(\mathbb{R}^d)$, with*

$$\|Tf\|_{\mathcal{W}(\mathcal{F}L^{r_2}, L^{r_1})} \lesssim \|\sigma\|_{M^{p,q}} \|f\|_{\mathcal{W}(\mathcal{F}L^{r_1}, L^{r_2})}.$$

The sharpness of the preceeding results can be derived as a special case of the following.

Proposition 3.16. *Let $1 \leq p, q, r_1, r_2, t_1, t_2 \leq \infty$. Consider the phase function $\Phi(x, \eta) = |x|^2/2 + x \cdot \eta$ which is tame and satisfies (2.10). Suppose the following estimates for the corresponding FIO T :*

$$(3.16) \quad \|Tf\|_{\mathcal{M}^{t_1, t_2}} \lesssim \|\sigma\|_{\mathcal{M}^{p, q}} \|f\|_{\mathcal{M}^{r_1, r_2}}, \quad \forall \sigma \in \mathcal{S}(\mathbb{R}^{2d}), \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Then

$$(3.17) \quad \frac{1}{r_1} - \frac{1}{t_2} \geq 1 - \frac{1}{p} - \frac{1}{q}, \quad \frac{1}{r_2} - \frac{1}{t_2} \geq 1 - \frac{1}{p} - \frac{1}{q}$$

and

$$(3.18) \quad q \leq \min(t_1, t_2, r'_1, r'_2).$$

Proof. For $\lambda > 0$, consider the family of FIOs T_λ , having phase function Φ and symbols $\sigma_\lambda = \varphi_{\lambda/\sqrt{2}} \otimes \varphi_{1/\lambda}$, with $\varphi(x) = e^{-\pi|x|^2}$ and $\varphi_\lambda(x) = \varphi(\lambda x)$. By assumption we have

$$(3.19) \quad \|T_\lambda \varphi_\lambda\|_{\mathcal{M}^{t_1, t_2}} \lesssim \|\sigma_\lambda\|_{\mathcal{M}^{p, q}} \|\varphi_\lambda\|_{\mathcal{M}^{r_1, r_2}}.$$

A straightforward computation shows that $T_\lambda \varphi_\lambda(x) = 2^{-d/2} e^{-\pi(\lambda^2 - i)|x|^2}$, so that, using (2.8) with $a = \lambda^2$ and $b = -1$, we obtain

$$\begin{aligned} \|T_\lambda \varphi_\lambda\|_{\mathcal{M}^{t_1, t_2}(\mathbb{R}^d)} &\asymp \frac{((\lambda^2 + 1)^2 + 1)^{\frac{d}{2}(\frac{1}{t_1} - \frac{1}{2})}}{\lambda^{\frac{d}{t_2}} (\lambda^2 (\lambda^2 + 1) + 1)^{\frac{d}{2}(\frac{1}{t_1} - \frac{1}{t_2})}} \\ &\asymp \begin{cases} \lambda^{-\frac{d}{t_2}}, & \lambda \rightarrow 0 \\ \lambda^{-d(1 - \frac{1}{t_2})}, & \lambda \rightarrow +\infty. \end{cases} \end{aligned}$$

From (2.9) we obtain

$$(3.20) \quad \begin{aligned} \|\sigma_\lambda\|_{\mathcal{M}^{p,q}(\mathbb{R}^{2d})} &\asymp \|\varphi_{\lambda/\sqrt{2}}\|_{\mathcal{M}^{p,q}(\mathbb{R}^d)} \|\varphi_{1/\lambda}\|_{\mathcal{M}^{p,q}(\mathbb{R}^d)} \\ &\asymp \begin{cases} \lambda^{-d/p} \lambda^{d/q'} = \lambda^{d(1-1/p-1/q)}, & \lambda \rightarrow 0 \\ \lambda^{-d/q'} \lambda^{d/p} = \lambda^{-d(1-1/p-1/q)}, & \lambda \rightarrow +\infty, \end{cases} \end{aligned}$$

whereas $\|\varphi_\lambda\|_{\mathcal{M}^{r_1,r_2}}$ depends on λ according to (2.14). Combining this with (3.19) we obtain for $\lambda \rightarrow 0$ the inequality

$$\frac{1}{r_1} - \frac{1}{t_2} \geq 1 - \frac{1}{p} - \frac{1}{q},$$

whereas letting $\lambda \rightarrow +\infty$ gives

$$\frac{1}{r_2} - \frac{1}{t_2} \geq 1 - \frac{1}{p} - \frac{1}{q}.$$

This proves (3.17).

In order to prove (3.18), define $h_\lambda(x) = h(x)e^{-\pi i \lambda |x|^2}$ for $\lambda \geq 1$ where $h \in C_c^\infty(\mathbb{R}^d) \setminus \{0\}$, $h \geq 0$, h even, and the parameter-dependent symbol $\sigma_\lambda = h \otimes h_\lambda$. Since σ_λ has support in a compact set independent of $\lambda \geq 1$, Lemma 2.3 (i) and [12, Lemma 4.2] give

$$(3.21) \quad \|\sigma_\lambda\|_{\mathcal{M}^{p,q}} \asymp \lambda^{d(\frac{1}{q}-\frac{1}{2})}, \quad \lambda \geq 1.$$

If we set $f_\lambda = \mathcal{F}^{-1}(\overline{h}_\lambda)$, then the operator with phase function Φ and symbol σ_λ acting on f_λ is

$$Tf_\lambda(x) = e^{\pi i |x|^2} h(x) \mathcal{F}^{-1} h^2(x).$$

Hence Tf_λ does not depend on λ , and we may choose h such that $1 \lesssim \|Tf_\lambda\|_{\mathcal{M}^{t_1,t_2}}$ for $\lambda \geq 1$. By Lemma 2.3 (ii) and [12, Lemma 4.2] we have

$$(3.22) \quad \|f_\lambda\|_{\mathcal{M}^{r_1,r_2}} \asymp \lambda^{d(\frac{1}{r_1}-\frac{1}{2})}.$$

Combining (3.21), (3.22) with the assumption (3.16) and the observation above we obtain

$$1 \lesssim \lambda^{d(\frac{1}{q}+\frac{1}{r_1}-1)}, \quad \lambda \geq 1,$$

which gives $q \leq r'_1$.

Next we define the symbol $\sigma_\lambda = \chi_n e^{-\pi i \lambda |\cdot|^2} \otimes h_\lambda$, where $\chi_n(x) = \chi(x/n)$, $n > 0$ is an integer, $\chi \in \mathcal{FC}_c^\infty(\mathbb{R}^d)$, χ real-valued and $\chi(0) = 1$. If $f = \mathcal{F}^{-1}h$ we obtain

$$Tf(x) = \chi_n(x) \mathcal{F}^{-1}(e^{-\pi i \lambda |\cdot|^2} h^2)(x).$$

Again [12, Lemma 4.2] gives

$$(3.23) \quad \|\mathcal{F}^{-1}(e^{-\pi i \lambda |\cdot|^2} h^2)\|_{L^{t_1}} \asymp \lambda^{d(\frac{1}{t_1}-\frac{1}{2})}, \quad \lambda \geq 1.$$

By means of dominated convergence we know that

$$\|(1 - \chi_n) \mathcal{F}^{-1}(e^{-\pi i \lambda |\cdot|^2} h^2)\|_{L^{t_1}} \leq \frac{1}{2} \|\mathcal{F}^{-1}(e^{-\pi i \lambda |\cdot|^2} h^2)\|_{L^{t_1}}$$

for $n \geq N$ where N is sufficiently large. Let $n \geq N$ be fixed. We have now

$$\begin{aligned} \lambda^{d(\frac{1}{t_1}-\frac{1}{2})} &\asymp \|\mathcal{F}^{-1}(e^{-\pi i \lambda |\cdot|^2} h^2)\|_{L^{t_1}} \\ &\leq 2 \left(\|\mathcal{F}^{-1}(e^{-\pi i \lambda |\cdot|^2} h^2)\|_{L^{t_1}} - \|(1 - \chi_n) \mathcal{F}^{-1}(e^{-\pi i \lambda |\cdot|^2} h^2)\|_{L^{t_1}} \right) \\ &\leq 2 \|\chi_n \mathcal{F}^{-1}(e^{-\pi i \lambda |\cdot|^2} h^2)\|_{L^{t_1}} = 2 \|Tf\|_{L^{t_1}}, \quad \lambda \geq 1. \end{aligned}$$

Because $\mathcal{F}\chi_n$ is supported in a fixed compact set for all n , $\mathcal{F}(Tf)$ is supported in a fixed compact set for all n and for all $\lambda \geq 1$. Thus Lemma 2.3 (ii) gives $\|Tf\|_{L^{t_1}} \asymp \|Tf\|_{\mathcal{M}^{t_1, t_2}}$. Since $\|f\|_{\mathcal{M}^{r_1, r_2}} \lesssim 1$, and (3.21) holds because n is fixed, when combined with the assumption (3.16), this gives

$$\lambda^{d(\frac{1}{t_1}-\frac{1}{2})} \lesssim \lambda^{d(\frac{1}{q}-\frac{1}{2})}, \quad \lambda \geq 1.$$

This implies $q \leq t_1$.

Next we note that (3.16) and (2.15) gives

$$(3.24) \quad \|\tilde{T}^* f\|_{W(\mathcal{F}L^{r'_1, L^{r'_2}})} \lesssim \|\sigma\|_{\mathcal{M}^{p, q}} \|f\|_{W(\mathcal{F}L^{t'_1, L^{t'_2}})}$$

where \tilde{T}^* is specified by (2.16). Let $\sigma_\lambda = h_\lambda \otimes h$, $f_\lambda = \mathcal{F}^{-1}(h_\lambda)$. Then

$$\tilde{T}^* f_\lambda(x) = h(x) \mathcal{F}^{-1}(e^{-\pi i |\cdot|^2} h^2)(x)$$

which implies $1 \lesssim \|\tilde{T}^* f_\lambda\|_{W(\mathcal{F}L^{r'_1, L^{r'_2}})}$ for all $\lambda \geq 1$.

As before we obtain

$$\|f_\lambda\|_{W(\mathcal{F}L^{t'_1, L^{t'_2}})} = \|h_\lambda\|_{\mathcal{M}^{t'_1, t'_2}} \asymp \lambda^{d(\frac{1}{t'_2}-\frac{1}{2})}, \quad \lambda \geq 1.$$

Combination with (3.24) and (3.21) now gives $q \leq t_2$.

Finally, in order to prove (3.18), it remains to verify $q \leq r'_2$. Let $\sigma_\lambda = h_{-\lambda} \otimes \chi_n$, $f = \mathcal{F}^{-1}h$. Then

$$\tilde{T}^* f(x) = \chi_n(x) \mathcal{F}^{-1}(e^{-\pi i (1+\lambda) |\cdot|^2} h^2)(x),$$

The same argument as above and Lemma 2.3 (i) give, for n sufficiently large (and fixed) and $\lambda \geq 1$,

$$\begin{aligned} \lambda^{d(\frac{1}{r'_2}-\frac{1}{2})} &\asymp \|\mathcal{F}^{-1}(e^{-\pi i (1+\lambda) |\cdot|^2} h^2)\|_{L^{r'_2}} \\ &\lesssim \|\chi_n \mathcal{F}^{-1}(e^{-\pi i (1+\lambda) |\cdot|^2} h^2)\|_{L^{r'_2}} \\ &= \|\widehat{\chi_n} * e^{-\pi i (1+\lambda) |\cdot|^2} h^2\|_{\mathcal{F}L^{r'_2}} \\ &= \|\widehat{\chi_n} * e^{-\pi i (1+\lambda) |\cdot|^2} h^2\|_{\mathcal{M}^{r'_1, r'_2}} \\ &= \|\tilde{T}^* f\|_{W(\mathcal{F}L^{r'_1, L^{r'_2}})}. \end{aligned}$$

The estimate (3.21) holds since n is fixed and by Lemma 2.3 (i) and [12, Lemma 4.2]

$$\|h_{-\lambda}\|_{\mathcal{M}^{p, q}} \asymp \|\mathcal{F}h_{-\lambda}\|_{L^q} = \|\mathcal{F}(\overline{h_\lambda})\|_{L^q} = \|\mathcal{F}h_\lambda\|_{L^q} \asymp \lambda^{d(\frac{1}{q}-\frac{1}{2})}, \quad \lambda \geq 1.$$

These considerations combined with (3.24) finally prove $q \leq r'_2$. \square

We apply the previous result to discuss the sharpness of Theorem 3.14. Indeed, if we choose $t_1 = r_2$ and $t_2 = r_1$, then (3.17) becomes

$$\frac{1}{p} + \frac{1}{q} \geq 1, \quad \frac{1}{r_2} - \frac{1}{r_1} \geq 1 - \frac{1}{p} - \frac{1}{q}$$

and (3.18) is $q \leq \min(r_1, r_2, r'_1, r'_2)$. If we assume $r_2 \leq r_1$, this can be rephrased as $q \leq \min(r_2, r'_1)$, and thus the conditions (3.13) are necessary under the assumption $r_2 \leq r_1$.

4. CONSEQUENCES FOR PSEUDODIFFERENTIAL OPERATORS

If we choose the phase function $\Phi(x, \eta) = x \cdot \eta$, the FIO reduces to a pseudodifferential operator in the Kohn–Nirenberg form. Boundedness results for pseudodifferential operators acting between modulation spaces are contained in many recent papers, see e.g. [5, 8, 12, 28, 29, 44, 45]. In particular, the action of a pseudodifferential operator between different modulation spaces was studied by Toft, and his result can be rephrased in our context as follows [44, Theorem 4.3].

Theorem 4.1. *Assume that $1 \leq p, q, r_1, t_1, r_2, t_2 \leq \infty$ satisfy*

$$(4.1) \quad 1/r_1 - 1/t_1 = 1/r_2 - 1/t_2 = 1 - 1/p - 1/q, \quad q \leq t_1, t_2 \leq p.$$

Then the pseudodifferential operator T , from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, having symbol $\sigma \in M^{p,q}(\mathbb{R}^{2d})$, extends uniquely to a bounded operator from $\mathcal{M}^{r_1, r_2}(\mathbb{R}^d)$ to $\mathcal{M}^{t_1, t_2}(\mathbb{R}^d)$, with the estimate

$$(4.2) \quad \|Tf\|_{\mathcal{M}^{t_1, t_2}} \lesssim \|\sigma\|_{M^{p,q}} \|f\|_{\mathcal{M}^{r_1, r_2}}.$$

We can now prove our main result concerning pseudodifferential operators, stated in the introduction.

Proof of Theorem 1.1. (i) *Sufficient conditions.* We observe that the assumptions of Theorem 3.9 are satisfied when $\Phi(x, \eta) = x \cdot \eta$ and the result follows immediately.

(ii) *Necessary conditions.* We now assume (1.3) and want to show that (1.4) and (1.5) hold.

For $\lambda > 0$, consider the families of pseudodifferential operators of Kohn–Nirenberg form T_λ , having phase function Φ and symbol $\sigma_\lambda = \varphi_{\lambda/\sqrt{2}} \otimes \varphi_{1/\lambda}$, with $\varphi(x) = e^{-\pi|x|^2}$ and $\varphi_\lambda(x) = \varphi(\lambda x)$. Observe that the behavior of these symbols is expressed by (3.20) in the proof of Proposition 3.16. By assumption we have

$$(4.3) \quad \|T_\lambda \varphi_\lambda\|_{\mathcal{M}^{t_1, t_2}} \lesssim \|\sigma_\lambda\|_{M^{p,q}} \|\varphi_\lambda\|_{\mathcal{M}^{r_1, r_2}}.$$

Since $T_\lambda \varphi_\lambda(x) = 2^{-d/2} e^{-\pi\lambda^2|x|^2}$ we obtain by (2.9)

$$(4.4) \quad \|T_\lambda \varphi_\lambda\|_{\mathcal{M}^{t_1, t_2}(\mathbb{R}^d)} \asymp \begin{cases} \lambda^{-\frac{d}{t_1}}, & \lambda \rightarrow 0, \\ \lambda^{-\frac{d}{t_2}}, & \lambda \rightarrow +\infty. \end{cases}$$

Combining (4.3), (4.4) with (3.20) and (2.9), we obtain for $\lambda \rightarrow 0$

$$\frac{1}{r_1} - \frac{1}{t_1} \geq 1 - \frac{1}{p} - \frac{1}{q},$$

whereas letting $\lambda \rightarrow +\infty$ gives

$$\frac{1}{r_2} - \frac{1}{t_2} \geq 1 - \frac{1}{p} - \frac{1}{q}.$$

This proves (1.4).

In order to prove (1.5) we set $h_\lambda(x) = h(x)e^{-\pi i \lambda |x|^2}$ for $\lambda \geq 1$, where $h \in C_c^\infty(\mathbb{R}^d) \setminus \{0\}$, $h \geq 0$, h even, and the parameter-dependent symbol $\sigma_\lambda = h \otimes h_\lambda$. If we set $f_\lambda = \mathcal{F}^{-1}(\bar{h}_\lambda)$, the operator with symbol σ_λ acting on f_λ is $Tf_\lambda(x) = h(x)\mathcal{F}^{-1}h^2(x)$. The same argument as in the proof of Proposition 3.16 gives $q \leq r'_1$.

If we switch quantization from the Kohn–Nirenberg $\sigma(x, D)$ to the Weyl quantization $\sigma^w(x, D)$, according to $\sigma(x, D) = a^w(x, D)$, then we have $\|\sigma\|_{M^{p,q}} \asymp \|a\|_{M^{p,q}}$ (cf. [44, Remark 1.5]). This fact, in combination with the Weyl quantization formula

$$\langle \sigma^w(x, D)f, g \rangle_{L^2} = \langle f, \bar{\sigma}^w(x, D)g \rangle_{L^2}, \quad f, g \in \mathcal{S}(\mathbb{R}^d),$$

allows us to conclude that the assumption (1.3) implies the dual result

$$\|Tf\|_{\mathcal{M}^{r'_1, r'_2}} \lesssim \|\sigma\|_{M^{p,q}} \|f\|_{\mathcal{M}^{t'_1, t'_2}}.$$

Reasoning as above we now obtain $q \leq t_1$, which proves $q \leq \min(t_1, r'_1)$.

It remains to show $q \leq \min(t_2, r'_2)$. We use the same arguments as in [12, Theorem 5.2]. Let us write down the details for the benefit of the reader. We consider the Weyl quantization $\sigma^w(x, D)$ and conjugate the operator $\sigma^w(x, D)$ with the Fourier transform. Then $\mathcal{F}^{-1}\sigma^w(x, D)\mathcal{F} = (\sigma \circ \chi)^w(x, D)$, where $\chi(x, \eta) = (\eta, -x)$. Moreover, the map $\sigma \mapsto \sigma \circ \chi$ is an isomorphism of $M^{p,q}$, so that (1.3) is equivalent to

$$\|Tf\|_{W(\mathcal{FL}^{t_1, L^{t_2}})} \lesssim \|\sigma\|_{M^{p,q}} \|f\|_{W(\mathcal{FL}^{r_1, L^{r_2}})} \quad \forall \sigma \in \mathcal{S}(\mathbb{R}^{2d}) \quad \forall f \in \mathcal{S}(\mathbb{R}^d),$$

where we switch back to the Kohn–Nirenberg form of the operator. Now we test the last estimate on the same families of symbols $\sigma_\lambda = h \otimes h_\lambda$ and functions $f_\lambda = \mathcal{F}^{-1}(\bar{h}_\lambda)$ as specified above. Observe that, since the functions f_λ have Fourier transforms supported in a fixed compact set, we have by Lemma 2.3 (i) $\|f_\lambda\|_{W(\mathcal{FL}^{r_1, L^{r_2}})} = \|\hat{f}_\lambda\|_{\mathcal{M}^{r_1, r_2}} \asymp \|f_\lambda\|_{L^{r_2}}$. Hence we get $q \leq r'_2$. By duality we finally obtain $q \leq t_2$. \square

Observe that, for $r_1 = r_2$ and $t_1 = t_2$, Theorem 1.1 gives the sharpness of Theorem 3.5.

Let us conclude by a comparison between our results and those of Theorem 4.1 ([44, Theorem 4.3]).

The index set corresponding to (1.4) and (1.5) is larger than the index set corresponding to (4.1). For simplicity, let us draw a picture for the particular case $t_i = r_i$, $i = 1, 2$. In this case (4.1) reduces to $q \leq t_1, t_2 \leq q' = p$, whereas (1.4) and (1.5) become $q \leq$

$\min(p', t_1, t_2, t'_1, t'_2)$. Projections of the set of points $(1/p, 1/q, 1/t_1, 1/t_2)$ that satisfy $q \leq t_1, t_2 \leq q' = p$ and $q \leq \min(p', t_1, t_2, t'_1, t'_2)$, respectively, onto the $(1/p, 1/q)$ -plane and onto the $(1/q, 1/t_i)$ -plane, $i = 1, 2$, are shown in Figures 1 and 2, respectively. We see that the range of exponents specified by (1.4) and (1.5) widens the exponents specified by (4.1) in the $(1/p, 1/q)$ -plane, whereas the range of exponents in the $(1/q, 1/t_i)$ -plane remains the same.

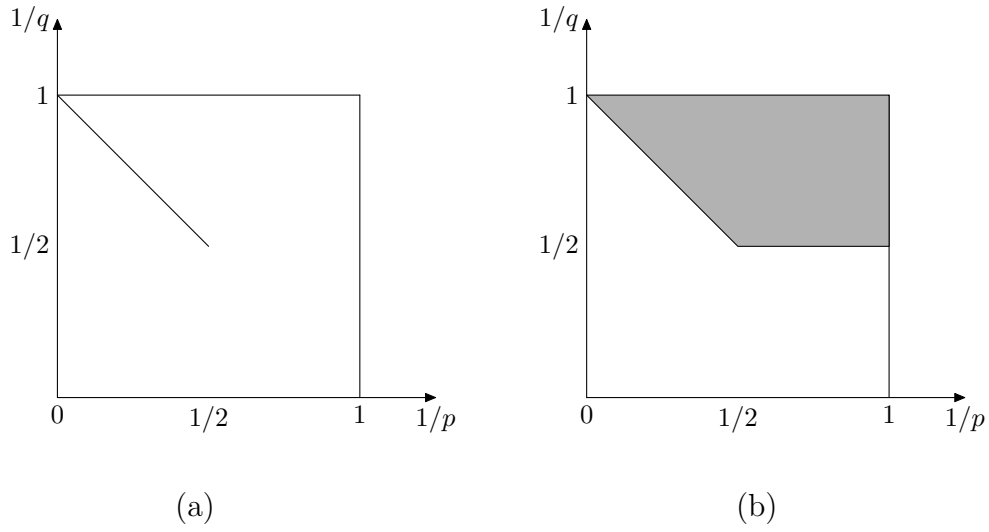


Figure 1. The range of exponents $(1/p, 1/q)$ when $t_i = r_i$, $i = 1, 2$:
(a) The range in (4.1), (b) The new range in (1.4) and (1.5).

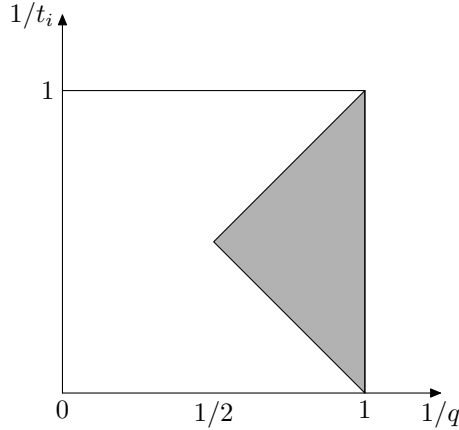


Figure 2. The range of exponents $(1/q, 1/r_i)$ when $t_i = r_i$, $i = 1, 2$:
(4.1), and (1.4), (1.5), coincide.

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